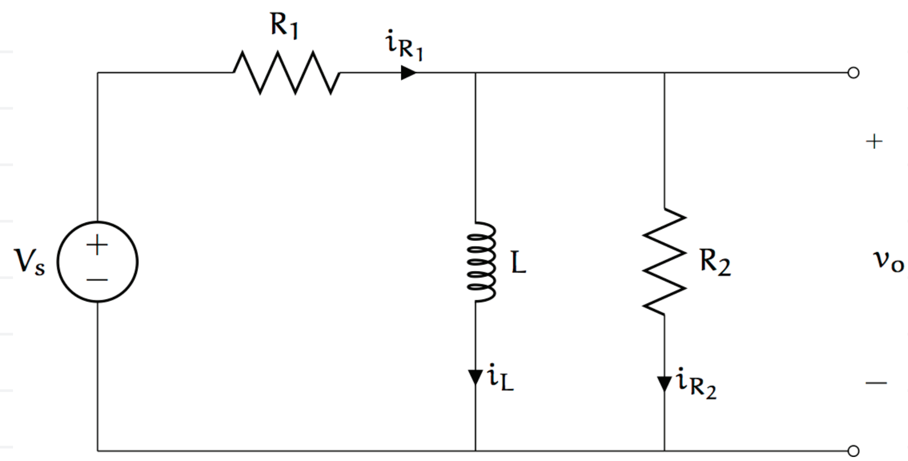
From Assignment #4

Special Problem #3. For the circuit diagram below, solve for the *steady state* $v_o(t)$ if $V_s(t) = A\ \sin\omega t$, where $A \in \mathbb{R}$ is a given amplitude and $\omega \in \mathbb{R}$ is a given angular frequency. Use impedances!



1. Signs. Already given

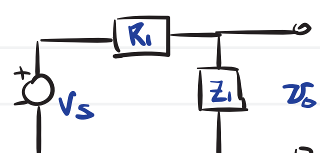
2. El. eq. $\quad v_{R_1} = i_{R_1} R_1 \quad v_{R_2} = i_{R_2} R_2 \quad v_L = i_L Z_L \quad (Z_L = j\omega L)$

3. KCL. $\quad i_{R_1} = i_L + i_{R_2}$

4. KVL. $\quad V_s = v_{R_1} + v_L \quad v_L = v_{R_2} \quad v_o = v_{R_2}$

5. Algebra. We could use the above equations, but we will take a shortcut. The generalized voltage divider can be applied if we combine the impedances of L and R_2:

$$Z_o = \frac{R_2 Z_L}{R_2 + Z_L} = \frac{j\omega R_2 L}{R_2 + j\omega L}.$$

The generalized voltage divider for  is

$$v_o = \frac{Z_o}{Z_o + R_1} V_s.$$

This is equivalent to the lengthier analysis that proceeds from the elemental eq's, KCL, + KVL.

All that remains is complex algebra:

$$v_o = \frac{\dfrac{j\omega R_2 L}{R_2 + j\omega L}}{\dfrac{j\omega R_2 L}{R_2 + j\omega L} + R_1} V_s$$

$$= \frac{j\omega R_2 L}{j\omega R_2 L + R_1(R_2 + j\omega L)} V_s$$

$$= \frac{j\omega R_2 L}{R_1 R_2 + j\omega L(R_1 + R_2)} V_s$$

$$= \frac{R_1 R_2 - j\omega L(R_1+R_2)}{R_1 R_2 - j\omega L(R_1+R_2)} \cdot \frac{j\omega R_2 L}{R_1 R_2 + j\omega L(R_1 + R_2)} V_s$$

$$= \frac{R_2(R_1+R_2)(\omega L)^2 + j\omega R_1 R_2^2 L}{(R_1R_2)^2 + (\omega L(R_1+R_2))^2} V_s$$

$$= \underbrace{\frac{\left((R_2(R_1+R_2)(\omega L)^2)^2 + (\omega R_1 R_2^2 L)^2\right)^{1/2}}{(R_1R_2)^2 + (\omega L(R_1+R_2))^2}}_{|Z_e|} e^{j\psi} V_s ,$$

where $\psi = \arctan \dfrac{\omega R_1 R_2^2 L}{R_2(R_1+R_2)(\omega L)^2} = \arctan \dfrac{R_1 R_2}{(R_1+R_2)\omega L}$.

Insert $V_s = A \sin\omega t = A\cos(\omega t - \pi/2) = A e^{-j\frac{\pi}{2}}$:

$$v_o = |Z_e|\, e^{j\psi}\left(A e^{-j\frac{\pi}{2}}\right)$$

$$= A|Z_e|\, e^{j(\psi - \pi/2)}$$

$$= A|Z_e| \cos(\omega t + \psi - \pi/2).$$