Practice test 2

Please e-mail me if you find errors in this or the solutions!

Instructions: I'd suggest trying this under test conditions.

1. Existence and Uniqueness theorem Perhaps: which of these DEs can we say have solutions? Are the solutions Unique? Where are they not Unique?
   $\frac{dy}{dt} = \frac{\sqrt{2-2y}}{\sqrt{2y-2}} = \sqrt{-1}$ so there's no y for which this works, so no solution. $\frac{dy}{dt} = \ln\left(\frac{1}{1+y^2} - 1\right)$ This has no value for y for which this works, so no solution. $\frac{dy}{dt} = \frac{t^2-4}{y^2-2y-3}$ As long as you're between the values of (-1,3) y is fine, so there's a rectangle of acceptable values. The partial derivative of the function only fails to have solutions at places where y can't have values, so the solutions are also unique. $\frac{dy}{dt} = \frac{y^2-2y-3}{t^2-4}$ As long as you're between the values of (-2,2) t is fine, so there's a rectangle of acceptable values. The partial derivative of the function never fails to have solutions so the solutions are also unique.

2. The Existence and Uniqueness theorems both hold for $\frac{dy}{dt} = f(t,y)$ and y_1, y_2 and $y(t)$ all solve the differential equation what can we say about $y(t)$ as t goes to ∞ if: $y_1 = -1$ and $y_2 = \frac{1}{t-1} - 1$ and $y(0) = -\frac{3}{2}$? $-2 < y(0) < -1$ so $y_2(0) < y(0) < y_1(0)$ and since they never cross: $y_2(t) < y(t) < y_1(t)$ or $\frac{1}{t-1} - 1 < y(t) < -1$ so $y(t)$ gets squeezed to -1 by y_1 as t goes to ∞.

3. Given that $\frac{dy}{dt} = 2\sqrt{|y|}$

   a. If $y(t) = 0$ for all t, then $\frac{dy}{dt} = 0$ and $2\sqrt{|y|} = 0$ for all t. Hence, the function that is constatly zero satisfies the differential equation.

   b. First, consider the case where $y > 0$. The differential equation reduces to $\frac{dy}{dt} = 2\sqrt{y}$, this is seperable, so we seperate, and integrate and get: $\sqrt{y} = t - c \quad y = (t-c)^2$, where c is some constant. This graph is the half of the parabola $t \geq c$. Now, if $y < 0$ then we have $\frac{dy}{dt} = 2\sqrt{-y}$, (where did that negative come from? Well, y is negative! So, to make it positive we have to have an explicit negative sign to make the argument of the root positive.) Now when we integrate we get $\sqrt{-y} = d - t \quad y = -(d-t)^2$, where d is some constant. This graph is the half of the parabola $t \leq d$.
   $y = \{-(d-t)^2, \quad t < d$
   $0, \quad d \leq t \leq c$
   $(t-c)^2, \quad t \geq c.$

   If you graph this out, it looks like $\frac{1}{2}$ of a downward parabola and $\frac{1}{2}$ of an upward parabola, seperated by a flat line at 0. This solves the differential equation, for any value c and d that you may care to use! But, c and d must be different parameters, since they come from different equations.

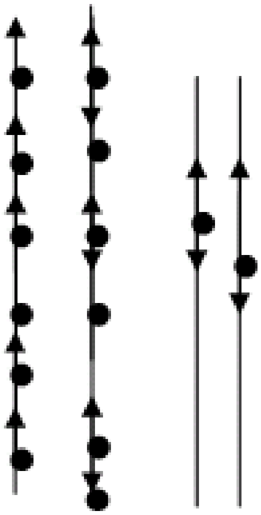
   c. $\frac{\partial y}{\partial t}$ doesn't exist along the t axis for $f(t,y) = \sqrt{|y|}$, so the uniqueness theorem doesn't apply.

4. Find f_x, f_y and f_z for the following functions:
   $f(x,y,z) = x\sin(x+y)\cos(x+z)$ so:

$f_x = \sin(x+y)\cos(x+z) + x\cos(x+y)\cos(x+z) - x\sin(x+y)\sin(x+z)$ and $f_y = x\cos(x+y)\cos(x+z)$ and $f_z = -x\sin(x+y)\sin(x+z)$

$f(x,y,z) = e^{x^2+y^2}$ means: $f_x = 2xe^{x^2+y^2}$, $f_y = 2ye^{x^2+y^2}$ and $f_z = 0$.

5. sketch the phase line and solutions for the following equations, label any equilibria as sources, sinks or nodes if they are neither a source nor a sink.
   $\frac{dy}{dt} = \cos^2 y$ $\frac{dy}{dt} = (1-y^2)\sin(2y)$ $\frac{dy}{dt} = y^3 - 8$ $\frac{dy}{dt} = y + 1$

   Documents/math322/phaselines.PNG 

6. Given the following conditions draw the possibile phase lines, why are there more than one? Draw the corresponding possible graphs of $f(y)$. These conditions force $y = 100$ to be equilibrium, because it's the only way for the function to change signs at that point. The slope for $y \in (0, 50)$ isn't specified so we get two different possibilities.

   Documents/math322/phaselines2.PNG 

7. Identify which of the following equations are linear. Identify which are forced and which are unforced. linear unforced: $\frac{dy}{dt} = yt^2$ linear forced: $\frac{dy}{dt} = t^2$ nonlinear: $\frac{dy}{dt} = y^2$ $\frac{dy}{dt} = y^2t^2$ $\frac{dy}{dt} = t^2\sin y + \cos t$

8. Justify the linearity principle. This is just proving that $y(t) = y_h + y_p$ for linear differential equations: $\frac{dy}{dt} = a(t)y + b(t)$ and we have that y_h solves $\frac{dy_h}{dt} = a(t)y_h$ and , $\frac{dy_p}{dt} = a(t)y_p + b(t)$ so:
   $\frac{d(y_h+y_p)}{dt} = a(t)(y_h + y_p) + b(t) = \frac{dy_p}{dt} + \frac{dy_h}{dt} = a(t)y_h + a(t)y_p + b(t) = a(t)(y_h + y_p) + b(t)$, which proves that if y_p solves the equation, so does $y(t) = y_h + y_p$.

9. Find the general solution for the following differential equation.
   $\frac{dy}{dt} = 7y + e^{6t}$ we use the linearity principle so $y(t) = y_h + y_p$, our $y_h = ke^{7t}$ and then our $y_p = \alpha e^{6t}$, pluggin in we get: $\frac{dy_p}{dt} = 7y_p + e^{6t} = \frac{d}{dt}(\alpha e^{6t}) = 7(\alpha e^{6t}) + e^{6t}$ $6\alpha e^{6t} = 7\alpha e^{6t} + e^{6t}$ $\alpha = -1$ so, plugging back in we get: $y(t) = y_h + y_p = ke^{7t} - e^{6t}$
   $\frac{dy}{dt} = 7y + e^{7t}$ we use the linearity principle,as before, but, note the degenerecy of the homogenious solution, so $y(t) = y_h + y_p$, our $y_h = ke^{7t}$ and then our $y_p = \alpha te^{7t}$, pluggin in we get: $\frac{dy_p}{dt} = 7y_p + e^{7t} = \frac{d}{dt}(\alpha te^{7t}) = 7(\alpha te^{7t}) + e^{7t}$ $7\alpha te^{7t} + \alpha e^{7t} = 7\alpha te^{7t} + e^{7t}$ $\alpha = 1$ so, plugging back in we get: $y(t) = y_h + y_p = ke^{7t} + te^{7t}$

10. Make a good guess of what the particular solution would look like for:
    $\frac{dy}{dt} = y + e^t + e^{-t} + t\sin(2t)$ $y_p = \alpha te^t + \beta e^{-t} + \gamma t\cos 2t + \delta t\sin 2t + \epsilon\cos 2t + \zeta\sin 2t$

11. Use integrating factors to find the solutions to the following differential equations.
    $\frac{dy}{dt} = -3\frac{y}{t} + 1 \Rightarrow \frac{dy}{dt} + 3\frac{y}{t} = 1$
    $b(t) = 1$ $g(t) = \frac{3}{t}$ $\mu(t) = e^{3\int\frac{dt}{t}} = e^{3\ln t} = e^{\ln t^3} = t^3$
    $y(t) = \frac{1}{\mu(t)}\int\mu(t)b(t)dt = \frac{1}{t^3}\int t^3 * 1dt = \frac{1}{t^3}(\frac{t^4}{4} + c) = \frac{t}{4} + \frac{c}{t^3}$

12. Use integrating factors to find the solutions to the following differnetial equations with the given initial conditions.
    $\frac{dy}{dt} = \frac{y}{t+1} + t^2 + t$ and $y(1) = 5$
    $\frac{dy}{dt} - \frac{y}{t+1} = t^2 + t$ so $b(t) = t^2 + t$ $g(t) = -\frac{1}{t+1}$ $\mu(t) = e^{\int\frac{dt}{t+1}} = t + 1$
    $y(t) = \frac{1}{\mu(t)}\int\mu(t)b(t)dt = (t+1)\int\frac{(t^2+t)dt}{t+1} = (t+1)\int tdt = (t+1)(\frac{t^2}{2} + c)$
    and employing the initial condition: $y(1) = 5 = (1+1)(\frac{1^2}{2} + c) = 1 + 2c$ $c = 2$ $y(t) = (t+1)(\frac{t^2}{2} + 2)$