Practice test 1 Solution

1. Simplify the following expressions

   (a) $\ln 16 - 2\ln 4 = 0$

   (b) $2\ln x^3 - \ln y^2 + \ln y^3 + 3\ln y = \ln\left(x^6y^4\right)$

   (c) $e^{4\ln x} - e^{3\ln x} = x^4 - x^3$

   (d) $\ln\left(e^{x^3}\right) - \ln\left(3e^x\right) = x^3 - x - \ln 3$

2. Solve for y:

   (a) $2y^2 + y = t + 5 \quad y = \frac{-1\pm\sqrt{1+8(t+5)}}{4}$

   (b) $\frac{y-4}{y} = t^2 + 3 \quad y - 4 = y(t^2+3) \quad y(-t^2-2) = 4 \quad y = \frac{-4}{t^2+2}$

   (c) $\frac{1}{y^2-1} = t^2 + 4 \quad y^2 - 1 = \frac{1}{t^2+4} \quad y = \pm\sqrt{1 + \frac{1}{t^2+4}}$

   (d) $\ln y = -\ln 2t \quad e^{\ln y} = e^{-\ln 2t} = y = \frac{1}{2t}$

   (e) $\ln y = -\ln t + \ln 3t^2 = \ln\left(\frac{3t^2}{t}\right) = \ln(3t) \quad y = 3t$

   (f) $\ln y - \ln(y+1) = \ln\left(\frac{y}{y+1}\right) = t^2 \quad \frac{y}{y+1} = e^{t^2} \quad y = e^{t^2}(y+1) \quad y(1-e^{t^2}) = e^{t^2} \quad y = \frac{e^{t^2}}{1-e^{t^2}}$

3. Find the derivative with respect to x of the following:

   (a) $\frac{d}{dx}\ln 2 = 0$

   (b) $\frac{d}{dx}e^{3x} = 3e^{3x}$

   (c) $\frac{d}{dx}\frac{1}{3x} = -\frac{1}{3x^2}$

   (d) $\frac{d}{dx}\frac{1}{x^2} = -\frac{2}{x^3}$

   (e) $\frac{d}{dx}\frac{1}{\sqrt{2x}} = -\frac{1}{2\sqrt{2x^3}}$

   (f) $\frac{d}{dx}\cos x = -\sin x$

   (g) $\frac{d}{dx}(e^x\sin(2x)) = e^x\sin(2x) + 2e^x\cos(2x)$

   (h) $\frac{d}{dx}\pi x = \pi$

   (i) $\frac{d}{dx}\left(\frac{2}{1+\sin x}\right) = -\frac{2\cos x}{(1+\sin x)^2}$

4. The next several exercises involve breaking apart a function into pieces that can be differentiated. Graph the function, and then look for the 'sharp points'. Find the derivative and then graph the derivative. Explain any discontinuities in the derivative.

5. (a) $\frac{d}{dx}\sqrt{|x|} = \begin{cases} \frac{1}{2\sqrt{x}} & x > 0 \\ -\frac{1}{2\sqrt{x}} & x < 0 \end{cases}$ (b) $\frac{d}{dx}|\cos x| = \begin{cases} \sin x & x \in (0,\pi),(2\pi,3\pi)\cdots \\ -\sin x & x \in (\pi,2\pi),(3\pi,4\pi)\cdots \end{cases}$

6. Integrate both sides to find $y(t)$:

   (a) $\frac{dy}{dt} = \sqrt{t} \quad y(t) = \sqrt{t^3} + C$

(b) $\frac{dy}{dt} = 2e^t \quad y(t) = 2e^t + C$

(c) $\frac{dy}{dt} = \frac{1}{t-1} \quad y(t) = \ln|t-1| + C$

(d) $\frac{dy}{dt} = \frac{1}{t^2} \quad y(t) = -\frac{1}{t} + C$

(e) $\frac{dy}{dt} = \sin(2t) \quad y(t) = -\frac{1}{2}\cos(2t)$

(f) $\frac{dy}{dt} = \sin t \cos t = \frac{1}{2}\sin(2t) \quad y(t) = \frac{1}{4}\cos(2t) + C$

(g) $\frac{dy}{dt} = \frac{1}{1+t^2} \quad y(t) = \tan^{-1} t + C$

(h) $\frac{dy}{dt} = \frac{3}{t(t-2)} = -\frac{3}{2t} + \frac{3}{2t-4} \quad y(t) = \frac{3}{2}(\ln|t-2| - \ln|t|) + C = \frac{3}{2}\ln\left|\frac{t-2}{t}\right| + C$

(i) $\frac{dy}{dt} = \frac{2}{(t-1)(t+2)} = -\frac{2}{3(t-1)} + \frac{2}{3(t+2)} \quad y(t) = \frac{2}{3}(\ln|t+2| - \ln|t-1|) + C = \frac{2}{3}\ln\left|\frac{t+2}{t-1}\right| + C$

7. You have a radioactive sample that loses 1/3 of it's radioactivity in an hour. What's it's half life?
$\frac{dN}{dt} = -kN$ gives us: $N = N_0 e^{-kt}$ and we can use $\frac{N}{N_0} = \frac{1}{3} = e^{-k(1hour)}$
So $\ln 3 = k$ and plugging that in to find the $t_{\frac{1}{2}}$ we get $\frac{1}{2} = e^{-ln3t}$ and $t = \frac{\ln 2}{\ln 3} = 0.63$ hours or 38 minutes.

8. 5 fish are introduced to a lake devoid of any other fish. After 6 months the population has tripled, after one year there are 9x as many fish, and so on. Write a population model based on this information. Do you think that your model will be reasonable after 20 years?
Since we can say that the population triples every six months we could write:
$\frac{dP}{dt} = 3P$ - where t is measured in 6 month intervals. Or we could just see that the population goes up by a factor of 9 and use the more logical t in units of years and have:
$\frac{dP}{dt} = 9P$
After 20 years you would have $P(20) = 5 * 9^{20} = 6x10^{19}$ fish, which is more than the Earth could hold, so the model seems a bit unreasonable.

9. You have a population of wombats in a given marsupial reserve. If there are fewer than 40 wombats they won't meet often enough to reproduce. But the area will only support 220 wombats. Design a logistic model that takes these parameters into account.
$\frac{dP}{dt} = \alpha P(P-40)(220-P)$ The equilibriam are at 0, 40 and 220. Below 40 and above 220 the population decreases, but between the two the population increases.

10. Given the following preditor-prey relationships:
$\frac{dP}{dt} = \alpha P - \beta P p$ and $\frac{dp}{dt} = \gamma p + \delta P p$
The α is the reproduction rate of the P, β is how many P die when they interact with p, because P are prey and p are preditors; the preditors increase when they interact with prey, and the prey decrease. γ is how fast the p, the preditors, increase, presumably, they can eat other prey as well, and δ is how much they benefit from a given prey (how often they catch it, or how much they need to eat).

11. You have the following differential equation $\frac{dy}{dt} = y^3 + y^2 - 2y$.

(a) For what values of y is $y(t)$ in equilibrium? 0, 1, -2

(b) For what values of y is $y(t)$ decreasing? below -2 and between 0 and 1

(c) For what values of y is $y(t)$ increasing? between -2 and 0 and above 1