Practice Final

1. Find $y(t)$ for the following differential equations, with given initial conditions:
   (a) $\frac{dy}{dt} = y^2$ and $y(1) = 1$.
   (b) $\frac{dy}{dt} = t^3 + t^2 + 1$ and $y(0) = 3$.
   (c) $\frac{dy}{dt} = t^2y^2 + ty + ty^2 + t^2y + y^2 + y$ and $y(0) = 0$
   (d) $\frac{dy}{dt} = t^2y^2 + ty + ty^2 + t^2y + y^2 + y$ and $y(0) = 4$
   (e) $\frac{dy}{dt} = \frac{t}{y+4}$ and $y(0) = -4$ What's wrong with this equation?
   (f) $\frac{dy}{dt} = \frac{t}{y+4}$ and $y(0) = 4$

2. Derive a logistic model for a species that dies out if it has fewer than 100, but once it has 100 its population is stable. If it has fewer animals than 200 it's population decreases. If the population is more than 200 it increases, as long as the population is less than 500. If the population is greater than 500, the population decreases.
   (a) Write out the logistic model.
   (b) Create a slope field, label the equilibria as stable or unstable.
   (c) Explain the significance of the missing solutions.
   (d) Draw a phase line.
   (e) Use the phase line to sketch $\frac{dP}{dt}$.

3. Given the following preditor-prey relationships:
   i. $\frac{dA}{dt} = \alpha A(A - m) - \beta AB$ and $\frac{dB}{dt} = -\gamma B + \delta AB$
   ii. $\frac{dC}{dt} = -\zeta C - \eta CD$ and $\frac{dD}{dt} = \epsilon D + \theta CD$
   Assuming all parameters are positive.
   (a) What do $\alpha, \beta, \gamma, \delta$ etc. represent?
   (b) What does m represent?
   (c) Which are preditor and which is prey?
   (d) If $\beta << \delta$ and $\eta \approx \theta$ what can we say about the relative sizes of preditors and prey?

4. If we have the equation $\frac{dy}{dt} = y^3t^2$, what are all of the solutions?

5. You have a radioactive sample that loses $\frac{7}{8}$ of it's radioactivity in a year. What's it's half life?

6. You have 5 liters of pure alcohol that you need to water down for a party. So you start drinking 1 liter per hour of whatever you've got. (Don't try this at home, 100% pure alcohol is dangerous.), you dump in enough Cheap American Beer (12 proof, or 6% alcohol) to keep the volume constant. How long would you have to do this untill you got a mix that's 80 proof (40% alcohol)?

7. a. Find the first two terms of the Taylor series of $\cos(2x)$ around $x = \frac{\pi}{4}$.

   b. Use this to find a linear approximation of $\cos(\frac{\pi}{2} + 0.1)$.

8. Use Euler's method to find an approximation for $t = [0, 1]$ if$\frac{dy}{dt} = t^2y^2$ with a $y(0) = 1$ if your step size is $\Delta t = 0.5$. Put it in tabular format. Columns would be: k, t, y, $\frac{dy}{dt}$.

9. Existence and Uniqueness theorem Which of these DEs can we prove have solutions? Are the solutions Unique? Where are they not Unique?
   $\frac{dy}{dt} = \frac{\sqrt{1+t}}{\sqrt{1-t}}$ $\quad \frac{dy}{dt} = \ln(\frac{t-\sqrt{y}}{\sqrt{1-y}})$ $\quad \frac{dy}{dt} = \ln(yt)$

10. Given that $\frac{dy}{dt} = \frac{y}{1+\sqrt{|-y|}}$

    (a) Show that $y(t) = 0$ is a solution.

    (b) Find all of the solutions. [Hint: when people did this on the homework they broke it up, but didn't think carefully about integration constants.]

    (c) Why doesn't this differntial equation contradict the uniqueness theorem?

11. Find f_x, f_y and f_z for the following functions:
    $f(x, y, z) = z\tan(x^2 + y^2)$ $\quad f(x, y, z) = e^{2xyz}$ $\quad f(x, y, z) = \ln x^2 + y$

12. sketch the phase line and solutions for the following equations, label any equilibria as sources, sinks or nodes if they are neither a source nor a sink.
    $\frac{dy}{dt} = \sin y$ $\quad \frac{dy}{dt} = (1 + y^2)$ $\quad \frac{dy}{dt} = -8$ $\quad \frac{dy}{dt} = y - 1$

13. Given the following conditions draw the phase line? Draw the corresponding possible graph of $f(y)$.
    $\frac{dy}{dt} = 0$ for $y = 0$
    $\frac{dy}{dt} < 0$ for $0 < y, 50$
    $\frac{dy}{dt} = 0$ for $y = 50$
    $\frac{dy}{dt} < 0$ for $100 > y > 50$
    $\frac{dy}{dt} > 0$ for $y > 100$

14. Identify which of the following equations are linear. Identify which are forced and which are unforced.
    $\frac{dv}{dx} = \sqrt{x}v$

    $\frac{dv}{dx} = \sqrt{v}x$

    $\frac{dv}{dx} = v\sin x$

    $\frac{dv}{dx} = x\sin v$

15. Find the general solution for the following differential equation.
$\frac{dy}{dt} = 3y + e^{3t}$ and $\frac{dy}{dt} = 2y + e^{3t}$

16. Make a good guess of what the particular solution would look like for (but don't solve):
$\frac{dy}{dt} = 3y + te^{3t} + t\sin(2t) + t^3 + t$

17. Use integrating factors to find the solutions to the following differential equations.
$\frac{dy}{dt} = -3\frac{y}{t+1} + t$

18. You have two different prey eaten by the same preditor. Differential equations for the two sets of preditor prey relations are provided (each presumes that there is only one preditor eating the prey, and one prey eaten by the preditor).
i. $\frac{dR}{dt} = R - RF$ $\quad \frac{dF}{dt} = -2F + RF$
ii. $\frac{dM}{dt} = 0.2M - 0.2MF$ $\quad \frac{dF}{dt} = -2F + 0.003LR$

   (a) Why is the coefficient of F in both $\frac{df}{dt}$ equations the same value (-2)?
   (b) Which prey is better at escaping?
   (c) Which prey is bigger? How do you know?
   (d) Find the equilibrium populations for i. and ii. seperately.
   (e) If the population of R starts off at 1 and the population of F starts off at 1, how do the populations change at this point?
   (f) If the population of M starts off at 1 and the population of F starts off at 1, how do the populations change at this point?
   (g) If you want to have an equation for $\frac{dF}{dt}$ that includes both what would it be?

19. (12 points) Given $\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$. Given that $b = 2\frac{kg}{s}$, $m = 10kg$ and $k = 4$,

   (a) Break this into two linear first order differential equations.
   (b) Find the equilibrium solution to these two linear first order differential equations.
   (c) Using the s method find the solution to the differential equation.

20. Consider $\frac{d^2y}{dt^2} = -9y$ with $y(0) = -3$ and $y'(0) = v(0) = 0$

   (a) Find $y(t)$ with the given initial conditions.
   (b) Graph the initial condition on a $y - v$ graph (where y is on the horizontal axis and v is on the vertical). And label this point 'I'.
   (c) Graph the solution to the differential equation, include the direction the solution goes.
   (d) A little damping is added, mathematically what does this look like for the differential equation?
   (e) Draw what the graph would look like now with a little bit of damping.
   (f) Now, instead of adding a little damping, add a lot of damping, what does the graph look like now?

21. Given $2A + 3B \rightarrow 2C$ and $A + C \rightarrow D$ and $D + 2B \rightarrow A$

    (a) What's the $\frac{dA}{dt}$?
    (b) What's the $\frac{dB}{dt}$?
    (c) What's the $\frac{dC}{dt}$?
    (d) What's the $\frac{dD}{dt}$?
    (e) If you start with no C or D will you ever get any D? (You have A and B)

22. Determine the vector field for $\frac{dx}{dt} = -x \quad \frac{dy}{dt} = -y$.

23. Determine the vector field for $\frac{dx}{dt} = x \quad \frac{dy}{dt} = y$.

24. Given the picture on the website draw x vs. t and y vs. t for A B and C.

25. Find $\mathbf{Y}(t)$ the solution to the set of differential equations: $\frac{dx}{dt} = 3x + y \quad \frac{dy}{dt} = -y$

26. (15 points) You have an RLC circuit with a variable capacitor (The value for C can change), and the differential equation: $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$

    (a) If L is fixed at 100H and R is fixed at 4Ω find the value for C that would give critical damping.
    (b) Assuming that the given graph gives critical damping, draw what would happen if you increased the capacitance of your capacitor.
    (c) Assuming that the given graph gives critical damping, draw what would happen if you decreased the capacitance of your capacitor.