

Practice Final Solutions

1. Find $y(t)$ for the following differential equations, with given initial conditions:

(a) $\frac{dy}{dt} = y^2$ and $y(1) = 1$.

$$\int \frac{dy}{y^2} = \int dt$$

$$-\frac{1}{y} = t + c \quad y = \frac{-1}{t+c} \quad y(1) = 1 = \frac{1}{1+c} \quad c = 0$$

$$y(t) = \frac{1}{t}$$

(b) $\frac{dy}{dt} = t^3 + t^2 + 1$ and $y(0) = 3$.

$$\int dy = y = \int (t^3 + t^2 + 1)dt = \frac{t^4}{4} + \frac{t^3}{3} + t + c \quad y(0) = 3 = 0 + 0 + 0 + c$$

$$y(t) = \frac{t^4}{4} + \frac{t^3}{3} + t + 3$$

(c) $\frac{dy}{dt} = t^2y^2 + ty + ty^2 + t^2y + y^2 + y$ and $y(0) = 0$

This is an example of hitting a missing solution right off, so $y(t) = 0$ is the solution.

(d) $\frac{dy}{dt} = t^2y^2 + ty + ty^2 + t^2y + y^2 + y$ and $y(0) = 4$

This one however eschews the missing solution, so we factor and get: $\frac{dy}{dt} = (y^2 + y)(t^2 + t + 1)$ and separate and use partial fractions:

$$\frac{1}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \text{ and cross multiply:}$$

$$\frac{1}{y^2+y} = \frac{A(y+1)+By}{y(y+1)} \text{ and then notice that the bottoms are equal so the top must be.}$$

$1 = Ay + A + By$ So then we take this equation and break it into equations with and without y :

$$1 = A \quad 0 = A + B \text{ which simplifies to } A = 1 \quad B = -1$$

$$\frac{1}{y^2+y} = \frac{1}{y} - \frac{1}{y+1} \text{ And then we get back to:}$$

$$\int \frac{dy}{y^2+y} = \int \frac{dy}{y} - \int \frac{dy}{y+1} = \int (t^2 + t + 1)dt = \frac{t^3}{3} + \frac{t^2}{2} + t + c$$

$$\ln|y| - \ln|y+1| = \ln\left|\frac{y}{y+1}\right| = \frac{t^3}{3} + \frac{t^2}{2} + t + c$$

$$\frac{y}{y+1} = k \exp\left(\frac{t^3}{3} + \frac{t^2}{2} + t\right) \text{ cross multiply and move:}$$

$$y = (y+1)ke^{stuff} = yke^{stuff} + ke^{stuff} \quad y(1 - ke^{stuff}) = ke^{stuff} \quad y = \frac{ke^{stuff}}{1+ke^{stuff}}$$

Then we plug in the initial conditions:

$$y(0) = 4 = \frac{k}{1+k} \quad k = \frac{4}{5}$$

$$y(t) = \frac{4 \exp\left(\frac{t^3}{3} + \frac{t^2}{2} + t\right)}{5 + 4 \exp\left(\frac{t^3}{3} + \frac{t^2}{2} + t\right)}$$

(e) $\frac{dy}{dt} = \frac{t}{y+4}$ and $y(0) = -4$ This is not allowed, because it would be dividing by zero.

(f) $\frac{dy}{dt} = \frac{t}{y+4}$ and $y(0) = 4$

$\int (y+4) = \int tdt = y^2 + 8y = t^2 + c$ We multiplied through by two, but now need the quadratic formula.

$$y = \frac{-8 \pm \sqrt{64 - 4(-t^2 - c)}}{2} \quad y(0) = 4 = -4 \pm \sqrt{16 + 0 + c} \quad c = 48$$

$y = -4 + \sqrt{64 + t^2}$ The (+) comes from the initial conditions, the (-) of the \pm wouldn't give us 4. Also, notice that we never pick up the forbidden value of $y = -4$ from this expression.

2. Derive a logistic model for a species that dies out if it has fewer than 100, but once it has 100 its population is stable. If it has fewer animals than 200 its population decreases. If the

population is more than 200 it increases, as long as the population is less than 500. If the population is greater than 500, the population decreases.

- (a) Write out the logistic model.
 $\frac{dP}{dt} = (P - 100)^2(200 - P)(P - 500)P$ We need the $(P - 100)^2$ term to give the node at $P = 100$.
- (b) Create a slope field, label the equilibria as stable or unstable. - See pictures on website.
- (c) Explain the significance of the missing solutions.
They are the equilibrium solutions, where the nodes are.
- (d) Draw a phase line.- See pictures on website.
- (e) Use the phase line to sketch $\frac{dP}{dt}$.- See pictures on website.

3. Given the following predator-prey relationships:

- i. $\frac{dA}{dt} = \alpha A(A - m) - \beta AB$ and $\frac{dB}{dt} = -\gamma B + \delta AB$
- ii. $\frac{dC}{dt} = -\zeta C - \eta CD$ and $\frac{dD}{dt} = \epsilon d + \theta CD$

Assuming all parameters are positive.

- (a) What do $\alpha, \beta, \gamma, \delta$ etc. represent?
 - α Reproduction rate for A
 - β How badly A does when being hunted by B
 - γ How fast B dies.
 - δ How much B benefits from hunting and eating A
 - ζ How fast C dies
 - η How badly C does when being hunted by D
 - ϵ How fast D reproduces
 - θ How well D does when hunting C .
- (b) What does m represent?
-The minimum population of A that makes it's population increase (in the absense of predation).
- (c) Which are predator and which is prey? A and C are prey. B and D are predators (although D probably has another source of food, and C seems to be dying off for some reason, like bad habitat).
- (d) If $\beta \ll \delta$ and $\eta \approx \theta$ what can we say about the relative sizes of predators and prey?
 A is much bigger than B while C and D are roughly the same size.

4. If we have the equation $\frac{dy}{dt} = y^3 t^2$, what are all of the solutions?

The first solution that should jump out at us is $y = 0$, a 'missing' solution. the rest is comparatively easy: $\int \frac{dy}{y^3} = \int t^2 dt$ $-\frac{1}{y^2} = \frac{t^3}{3} + c_1$

$y = \frac{1}{\sqrt{c+t^3}}, 0$ In this case the $y = 0$ solution isn't at all obvious at the end, which is why you need to find it at the beginning.

5. You have a radioactive sample that loses $\frac{7}{8}$ of it's radioactivity in a year. What's it's half life? Since the sample is losing $\frac{7}{8}$ of it's radioactivity, that means that after 1 year it has $\frac{1}{8}$ of what it started with, which is 3 half lives, which means $3\tau_{\frac{1}{2}} = 1$ year, so $\tau_{\frac{1}{2}} = 4$ months.

6. You have 5 liters of pure alcohol that you need to water down for a party. So you start drinking 1 liter per hour of whatever you've got. (Don't try this at home, 100% pure alcohol is dangerous.), you dump in enough Cheap American Beer (12 proof, or 6% alcohol) to keep the volume constant. How long would you have to do this until you got a mix that's 80 proof (40% alcohol)? The 1 liter per hour means that we need to subtract off 1 liter * concentration of alcohol. We are adding 1 liter per hour of cheap american beer which means we're adding $0.06/5$. (The divided by 5 was in the wrong place in my notes.) So that gives us $\frac{dC}{dt} = -1 * C + \frac{0.06}{5}$ Which is separable: $\int \frac{dC}{0.003-C} = \int dt$
 $-\ln|0.003 - C| = t + c$ $C = 0.003 + ke^{-t}$ The concentration of alcohol at $t = 0$ is 1. ($C(0) = 1$), $k = 0.997$. $C(t) = 0.003 + 0.997e^{-t}$ $0.4 = 0.003 + 0.997e^{-t}$ $t = -\ln \frac{0.397}{0.997}$ And without a calculator, that's as far as you can go! (But the answer is 0.92 hours).
7. a. Find the first two terms of the Taylor series of $\cos(2x)$ around $x = \frac{\pi}{4}$.
 $f'(x) = -2 \sin(2x)$ $f''(x) = -4 \cos(2x)$
 $f(\frac{\pi}{4}) = \cos(2\frac{\pi}{4}) = 0$ $f'(\frac{\pi}{4}) = -2 \sin(2\frac{\pi}{4}) = -2$ $f''(x) = -4 \cos(2\frac{\pi}{4}) = 0$
 $f(x) = 0 - 2(x - \frac{\pi}{4}) + 0 = -2x + \frac{\pi}{2}$
- b. Use this to find a linear approximation of $\cos(\frac{\pi}{2} + 0.1)$.
 $x = \frac{\pi}{4} + 0.05$ So we plug that in to part a.
 $f(\frac{\pi}{4} + 0.05) = -2(\frac{\pi}{4} + 0.05 - \frac{\pi}{4}) = -2(0.05) = -0.1$
8. Use Euler's method to find an approximation for $t = [0, 1]$ if $\frac{dy}{dt} = t^2 y^2$ with a $y(0) = 1$ if your step size is $\Delta t = 0.5$. Put it in tabular format. Columns would be: k , t , y , $\frac{dy}{dt}$. See picture of solutions.
9. Existence and Uniqueness theorem Which of these DEs can we prove have solutions? Are the solutions Unique? Where are they not Unique?
 $\frac{dy}{dt} = \frac{\sqrt{1+t}}{\sqrt{1-t}}$ This one works for $t \in [-1, 1)$ and for all y .
 $\frac{dy}{dt} = \ln(\frac{t-\sqrt{y}}{\sqrt{1-y}})$ If $t > 1$ then $y \in [0, 1)$
 $\frac{dy}{dt} = \ln(yt)$ Both t and y must be greater than zero, or both must be less than 0.
10. The Existence and Uniqueness theorems both hold for $\frac{dy}{dt} = f(t, y)$ and y_1, y_2 and $y(t)$ all solve the differential equation what can we say about $y(t)$ as t goes to ∞ if: $y_1 = t$ and $y_2 = 1 + t^2$
- (a) If $y(0) = 0$? - Then $y(0) = y_1(0)$ so $y(t) = y_1(t) = t$ So $y(t) = t$
- (b) If $y(0) = 3$? - Then $y(0) > y_2(0)$ so $y(t) > y_2(t)$ so $y(t)$ is always greater so it goes to infinity as t goes to $\pm\infty$
- (c) If $y(0) = -\frac{3}{2}$? - Then $y(0) < y_1(0)$ so $y(t) < y_1(t)$ so $y(t)$ is always less so it goes to negative infinity as t goes to $-\infty$.
11. The Existence and Uniqueness theorems both hold for $\frac{dy}{dt} = f(t, y)$ and y_1, y_2 and $y(t)$ all solve the differential equation what can we say about $y(t)$ as t goes to ∞ if: $y_1 = t$ and $y_2 = 1 + t^2$
- (a) If $y(0) = 0$?
- (b) If $y(0) = 3$?

(c) If $y(0) = -\frac{3}{2}$?

12. Given that $\frac{dy}{dt} = \frac{y}{1+\sqrt{|-y|}}$

(a) Show that $y(t) = 0$ is a solution. – Plug it in, the equation is satisfied.

(b) Find all of the solutions. [Hint: when people did this on the homework they broke it up, but didn't think carefully about integration constants.] For $y > 0$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{y}{1+\sqrt{y}} & \int \frac{1+\sqrt{y}}{y} dt &= \int \frac{1}{y} + \frac{1}{\sqrt{y}} \\ &= \ln y + 2\sqrt{y} & &= t + c\end{aligned}$$

For $y < 0$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{y}{1+\sqrt{-y}} & \int \frac{1+\sqrt{-y}}{y} dt &= \int \frac{1}{y} + \frac{1}{\sqrt{-y}} \\ &= \ln -y - 2\sqrt{-y} & &= t + d\end{aligned}$$

The d is because it's a different integration constant in the same problem

For $y = 0$ then $y(t) = 0$ for all time. So, we can have the solutions cross, which seems strange.

(c) Why doesn't this differential equation contradict the uniqueness theorem? Because the partial of $\frac{y}{1+\sqrt{|y|}}$ is discontinuous at $y = 0$.

13. Find f_x , f_y and f_z for the following functions:

$$f(x, y, z) = z \tan(x^2 + y^2) \quad f_z = \tan(x^2 + y^2) \quad f_x = 2xz \sec^2(x^2 + y^2) \quad f_y = 2yz \sec^2(x^2 + y^2)$$

$$f(x, y, z) = e^{2xyz} \quad f_x = 2xe^{2xyz} \quad f_y = 2ye^{2xyz} \quad f_z = 2ze^{2xyz}$$

$$f(x, y, z) = \ln x^2 + y \quad f_x = \frac{2x}{x^2+y} \quad f_y = \frac{1}{x^2+y} \quad f_z = 0$$

14. sketch the phase line and solutions for the following equations, label any equilibria as sources, sinks or nodes if they are neither a source nor a sink.

$$\frac{dy}{dt} = \sin y \quad \frac{dy}{dt} = (1 + y^2) \quad \frac{dy}{dt} = -8 \quad \frac{dy}{dt} = y - 1 \quad \text{See pictures on the website.}$$

15. Given the following conditions draw the phase line? Draw the corresponding possible graph of $f(y)$. See pictures on the website. $\frac{dy}{dt} = 0$ for $y = 0$

$$\frac{dy}{dt} < 0 \text{ for } 0 < y, 50$$

$$\frac{dy}{dt} = 0 \text{ for } y = 50$$

$$\frac{dy}{dt} < 0 \text{ for } 100 > y > 50$$

$$\frac{dy}{dt} > 0 \text{ for } y > 100$$

16. Identify which of the following equations are linear. Identify which are forced and which are unforced.

$$\frac{dv}{dx} = \sqrt{x}v \text{ linear}$$

$$\frac{dv}{dx} = \sqrt{v}x \text{ nonlinear}$$

$$\frac{dv}{dx} = v \sin x \text{ linear}$$

$$\frac{dv}{dx} = x \sin v \text{ nonlinear All are unforced.}$$

17. Find the general solution for the following differential equation.

$$\frac{dy}{dt} = 3y + e^{3t}$$

$$y_h = ke^{3t} \quad y_p = ate^{3t}$$

$\frac{dy_p}{dt} = 3y_p + e^{3t} = \frac{d}{dt}\alpha te^{3t} = 3(\alpha te^{3t}) + e^{3t} = 3\alpha te^{3t} + \alpha e^{3t} = 3\alpha te^{3t} + e^{3t}$ Then separate out into different equations:

$$3\alpha t = 3\alpha t \quad \alpha = 1 \quad \text{So } \alpha = 1$$

$$y_p = te^{3t} \quad \text{so} \quad y = y_h + y_p = ke^{3t} + te^{3t}$$

and $\frac{dy}{dt} = 2y + e^{3t}$ $y_h = ke^{2t}$ $y_p = \beta e^{3t}$

$$\frac{dy_p}{dt} = 2y_p + e^{3t} = \frac{d}{dt}\beta e^{3t} = 2(\beta e^{3t}) + e^{3t} = 3\beta e^{3t} = 2\beta e^{3t} + e^{3t}$$

So $\beta = 1$

$$y_p = e^{3t} \quad \text{so} \quad y = y_h + y_p = ke^{2t} + e^{3t}$$

18. Make a good guess of what the particular solution would look like for (but don't solve):

$$\frac{dy}{dt} = 3y + te^{3t} + t \sin(2t) + t^3 + t$$

$$y_p = \alpha te^{3t} + \beta t \sin(2t) + \gamma \cos(2t) + \eta t \sin(2t) + \theta \cos(2t) + At^3 + Bt^2 + Ct + D.$$

19. Use integrating factors to find the solutions to the following differential equations.

$$\frac{dy}{dt} = -3\frac{y}{t+1} + t$$

$$\mu(t) = e^{\int 3\frac{dt}{t+1}} = e^{3\ln t+1} = e^{(t+1)=(t+1)^3}$$

And plug in:

$$y(t) = \frac{1}{(t+1)^3} \int (t+1)^3 t dt = \frac{\frac{t^5}{5} + 3\frac{t^4}{4} + t^3 + \frac{t^2}{2} + c}{(t+1)^3}$$

20. You have two different prey eaten by the same predator. Differential equations for the two sets of predator-prey relations are provided (each presumes that there is only one predator eating the prey, and one prey eaten by the predator).

i. $\frac{dR}{dt} = R - RF$ $\frac{dF}{dt} = -2F + RF$

ii. $\frac{dM}{dt} = 0.2M - 0.2MF$ $\frac{dF}{dt} = -2F + 0.003MF$

- (a) Why is the coefficient of F in both $\frac{df}{dt}$ equations the same value (-2)?

They die at the same rate regardless of what they're eating.

- (b) Which prey is better at escaping?

M , the interactions aren't as bad for the M .

- (c) Which prey is bigger? How do you know?

The F benefit more from interactions with the R than the M , so the R must be bigger (even accounting for the difference in the success rate of the hunting).

- (d) Find the equilibrium populations for i. and ii. separately.

i. $R = 2$ $F = 1$ ii. $F = 1$ $M = 2/0.003 = 666$.

- (e) If the population of R starts off at 1 and the population of F starts off at 1, how do the populations change at this point?

i. $\frac{dR}{dt} = R - RF = 1 - 1 = 0$ $\frac{dF}{dt} = -2(1) + (1)(1) = -1$ So R stays the same and F decreases.

- (f) If the population of M starts off at 1 and the population of F starts off at 1, how do the populations change at this point? ii. $\frac{dM}{dt} = 0.2M - 0.2MF = 0.2(1) - 0.2(1)(1) = 0$

$\frac{dF}{dt} = -2F + 0.003MF = -2(1) + 0.003(1)(1) = -1.997$ So the M is stable and the

F are decreasing. It's a little counterintuitive, but the F population is what makes M and R increase or decrease, not the R or M population, and the contrapositive is also true.

- (g) If you want to have an equation for $\frac{dF}{dt}$ that includes both what would it be?
 $\frac{dF}{dt} = -2F + RF + 0.003MF$

21. (12 points) Given $\frac{d^2y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{k}{m}y = 0$. Given that $b = 2\frac{kg}{s}$, $m = 10kg$ and $k = 4$,

- (a) Break this into two linear first order differential equations. $\frac{dy}{dt} = v$ $\frac{dv}{dt} = -\frac{b}{m}v + \frac{k}{m}y$.
 (b) Find the equilibrium solution to these two linear first order differential equations. Both have to be zero, it's the only equilibrium available!
 (c) Using the s method find the solution to the differential equation. $s^2 + 0.2s + 0.4 = 0$
 $s = \frac{-0.2 \pm \sqrt{0.04 - 1.6}}{2}$
 $y(t) = e^{-0.1t}[\cos(\sqrt{1.59}t) + i \sin(\sqrt{1.59}t)]$

22. Consider $\frac{d^2y}{dt^2} = -9y$ with $y(0) = -3$ and $y'(0) = v(0) = 0$

- (a) Find $y(t)$ with the given initial conditions. - see pictures.
 (b) Graph the initial condition on a $y - v$ graph (where y is on the horizontal axis and v is on the vertical). And label this point 'I'.
 (c) Graph the solution to the differential equation, include the direction the solution goes.
 (d) A little damping is added, mathematically what does this look like for the differential equation?
 (e) Draw what the graph would look like now with a little bit of damping.
 (f) Now, instead of adding a little damping, add a lot of damping, what does the graph look like now?

23. Given $2A + 3B \rightarrow 2C$ and $A + C \rightarrow D$ and $D + 2B \rightarrow A$

- (a) What's the $\frac{dA}{dt} = -2\alpha A^2 B^3 - \beta AC + \gamma DB^2$
 (b) What's the $\frac{dB}{dt} = -3\alpha A^2 B^3 - 2\gamma DB^2$
 (c) What's the $\frac{dC}{dt} = +\alpha A^2 B^3 - \beta AC$
 (d) What's the $\frac{dD}{dt} = -\beta AC - \gamma DB^2$
 (e) If you start with no C or D will you ever get any D ? (You have A and B) Sure! The A and B combine to make C and then A and C combine to make D !

24. Determine the vector field for $\frac{dx}{dt} = -x$ $\frac{dy}{dt} = -y$. See pic

25. Determine the vector field for $\frac{dx}{dt} = x$ $\frac{dy}{dt} = y$. See pic

26. Find $\mathbf{Y}(t)$ the solution to the set of differential equations: $\frac{dx}{dt} = 3x + y$ $\frac{dy}{dt} = -y$
 $y = k_1 e^{-t}$ And plug that in: $\frac{dx}{dt} = 3x + y = 3x + k_1 e^{-t}$
 $x_h = k_2 e^{3t}$ $x_p = \alpha e^{-t}$

$$\begin{aligned} \frac{dx}{dt} &= 3x + y = 3x + k_1 e^{-t} = \frac{d}{dt} \alpha e^{-t} = -\alpha e^{-t} = 3(\alpha e^{-t}) + k_1 e^{-t} \\ \alpha &= -\frac{k_1}{4} \\ x_p &= -\frac{k_1}{4} e^{-t} \\ x(t) &= x_h + x_p = k_2 e^{3t} - \frac{k_1}{4} e^{-t} \end{aligned}$$

27. (15 points) You have an RLC circuit with a variable capacitor (The value for C can change), and the differential equation: $\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = 0$

(a) If L is fixed at 100H and R is fixed at 4Ω find the value for C that would give critical damping.

$$\frac{R^2}{L^2} = \frac{4}{LC} \quad C = \frac{4L}{R^2} = \frac{400}{4^2} = 25 \text{ Farads}$$

(b) Assuming that the given graph gives critical damping, draw what would happen if you increased the capacitance of your capacitor. - Increasing the capacitance would effectively increase the damping, making an overdamped situation.

(c) Assuming that the given graph gives critical damping, draw what would happen if you decreased the capacitance of your capacitor. - decreasing the capacitance would effectively decrease the damping, making an underdamped situation.